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EQUIVALENCE OF SET- AND BAG-VALUED ORBITS

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ABSTRACT

It is shown that to check the equivalence of two set-valued or bag-valued orbits over the rationals it suffices to verify equality of N initial terms of the orbits where N depends only on the dimension of the orbits. An explicit form for N is given.

Keywords: Orbits, equivalence problem

1. Introduction

An *orbit* of dimension k over a field F is a sequence of the form $(\mathbf{M}^n\mathbf{c})_{n=0}^{\infty}$ where $\mathbf{M}$ is a $k \times k$-matrix and $\mathbf{c}$ is a k-vector, both over F. Orbits of this kind became quite well-known mainly because of a famous problem, called the Orbit Problem. The problem asks whether or not a given vector appears as a term of a given orbit. Originally stated by Harrison in [4] (for computable fields F and in connection with linear sequential machines), it was finally shown to be decidable in polynomial time for $F = \mathbb{Q}$ by Kannan and Lipton [5]. The Kannan–Lipton Algorithm quickly established itself as a fundamental algorithmic paradigm.

While the solution of the Orbit Problem is highly nontrivial, the equivalence of two given orbits $(\mathbf{M}^n\mathbf{c})_{n=0}^{\infty}$ and $(\mathbf{N}^n\mathbf{d})_{n=0}^{\infty}$ of the same dimension is quite easily understood, and certainly solvable in polynomial time when $F = \mathbb{Q}$. This follows because, by the Cayley–Hamilton Theorem applied to the direct sum $\mathbf{M} \oplus \mathbf{N}$, the difference $\mathbf{f}_n = \mathbf{M}^n\mathbf{c} - \mathbf{N}^n\mathbf{d}$ satisfies the linear homogeneous recurrence with constant coefficients (LHRCC in short)

$$\mathbf{f}_n = c_1\mathbf{f}_{n-1} + c_2\mathbf{f}_{n-2} + \cdots + c_{2k}\mathbf{f}_{n-2k} \quad \text{for} \quad n \geq 2k$$

of order $2k$, where $1, -c_1, -c_2, \ldots, -c_{2k}$ are the coefficients of the (monic) characteristic polynomial of $\mathbf{M} \oplus \mathbf{N}$. Thus, to check the equivalence of the orbits, it suffices to do this for the first $2k$ terms.

We are interested in *set-valued orbits* of the form

$$\big(\{\mathbf{M}^n\mathbf{c}_1, \ldots, \mathbf{M}^n\mathbf{c}_m\}\big)_{n=0}^{\infty},$$

and their equivalence. This generalization of orbits appears to be unrelated to earlier extensions, such as the extension to two commuting matrices in [1].