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ON SQUARE-FREE PERMUTATIONS

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ABSTRACT

A permutation is square-free if it does not contain two consecutive factors of length more than one that coincide in the reduced form (as patterns). We prove that the number of square-free permutations of length n is $n^{n(1-\varepsilon_n)}$ where $\varepsilon_n \to 0$ when $n \to \infty$.

A permutation of length n is crucial with respect to squares if it avoids squares but any extension of it to the right, to a permutation of length $n+1$, contains a square. A permutation is maximal with respect to squares if both the permutation and its reverse are crucial with respect to squares. We prove that there exist crucial permutations with respect to squares of any length at least 7, and there exist maximal permutations with respect to squares of odd lengths $8k+1$, $8k+5$, $8k+7$ for $k \geq 1$.

Keywords: Square freeness, consecutive pattern, enumeration, crucial word, maximal word, permutation

1. Introduction

A *square* in a word is a classical concept in combinatorics on words meaning two equal consecutive factors in the word. For example, the word 213413413 contains the square 134134, whereas the word 2141231 is square-free. It was first established by Thue [8] that there are arbitrary long square-free words over 3 (or more) letter alphabets,